ON THE EXISTENCE OF CHAOTIC SOLUTIONS IN DYNAMIC LINEAR PROGRAMMING*

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ABSTRUCT

This paper reports our result demonstrating that chaos may emerge as a solution to a dynamic linear programming (LP) problem. In what follows, we will first discuss the basic motivation of our research in Section 1 and then set up the specific LP problem that we deal with in Section 2. We will state our result in the form of a theorem in Section 3. In Section 4, we will make concluding remarks concerning the direction of future research.

1. INTRODUCTION

It has been well-known that dynamic programming can be treated in the standard LP framework by adding a time structure (Dorfman, Samuelson and Solow, 1958). In order to demonstrate the existence of chaotic solutions to such a problem, we need to focus on the case in which the solutions to an LP problem can be described by an autonomous system; chaos is a phenomenon that appears in autonomous system.

For this reason, it is necessary to work with an infinite-time horizon LP model. Now, think of the following LP problem:

(1.1)

$$\begin{cases} \underset{(x_1, x_2, \cdots)\geq 0}{\text{maximize}} & \sum_{t=1}^{\infty} \rho^{t-1} p' x_t \\ \text{subject to} & \begin{bmatrix} B & 0 & 0 & 0 & \cdots \\ -A & B & 0 & 0 & \cdots \\ 0 & -A & B & 0 & \cdots \\ 0 & 0 & -A & B & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix} \leq \begin{bmatrix} Ax+d \\ d \\ d \\ \vdots \\ \vdots \end{bmatrix}. \end{cases}$$

In the above problem, the discount factor ρ is a number between 0 and 1, A and B are m$\times$n matrices of non-negative components, d and p are n$\times$1 matrices of

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non-negative components, and p' is the transpose of p. The intended interpretation of this problem is to maximize the objective function $\sum_{t=1}^{\infty} \rho^{t-1} p' x_t$, which is the discounted sum of $p' x_t$ over the time periods t=1,2,..., under the recursive constraints $Bx_t \leq Ax_{t-1} + d$ with the initial condition $x_0 = x$.

By the Bellman principle (see Bellman, 1957, and Bellman and Kalaba, 1965), the solutions to problem (1.1) can be described by a binary relation F. That is to say, $(x_1^*, x_2^*, ...)$ solves (1.1) if and only if $x_0^* = x$ and

$$(1.2) \qquad x_t^* \in F(x_{t-1}^*)$$

for t=1,2,..., ∞. In this sense, we may call the binary relation F an *optimal program.*[1] Because of its linear structure, the maximization problem, (1.1), generally has multiple solutions. As a result, the binary relation describing the solutions, F, is a set-valued function.

The question that we face is whether or not such an optimal program can be a chaotic dynamical system. In order to deal with this issue, we need to answer the following two specific questions. (i) Under what condition, the optimal program, (1.2), is in fact a dynamical system of the standard sense, described by a single-valued function instead of the set-valued function F? (ii) Under what condition the resulting dynamical system is chaotic? In order to deal with these questions, we construct a simple LP problem with an infinite-time horizon. We then derive conditions under which the solutions are described by a chaotic dynamical system.

2. A SIMPLE DYNAMIC LP PROBLEM

Think of the following LP problem with parameters $a_{11} > 0, a_{12} > 0, a_{21} > 0, a_{22} > 0, k > 0$ and ρ, 0< ρ <1.

(2.1)

$$\begin{cases} \underset{(c_1, k_1, c_2, k_2, \cdots) \geq 0}{\text{maximize}} & \sum_{t=1}^{\infty} \rho^{t-1} c_t \\ \text{subject to} & \text{(i) } a_{11}c_t + a_{12}k_t \leq 1 \qquad t = 1, 2, \cdots \\ & \text{(ii) } a_{21}c_t + a_{22}k_t \leq k_{t-1} \qquad t = 1, 2, \cdots \\ & \text{(iii) } k_0 = k. \end{cases}$$

As is noted above, the solutions to problem (2.1) can be described by a generalized dynamical system. To this end, for each (k_{t-1}, k_t), define $c(k_{t-1}, k_t)$ as the maximum value of $c_t \geq 0$ satisfying conditions (i) and (ii) of (2.1).

Proposition1: *For each $k \geq 0$, there is a non-empty subset of R_+, H(k), such that*

[1] If (1.1) is of a finite-time horizon, the optimal program depends on time, and the solutions to that problem can be written as $x_t \in F_t(x_{t-1})$.

if $(c_1, k_1, c_2, k_2, \cdots)$ *is a solution to (2.1), then it holds that*

(2.2) $k_t \in H(k_{t-1}),\ t = 1, 2, \cdots, \infty,$

with $k_0 = x$ *and that*

(2.3) $c_t = c(k_{t-1}, k_t)$.

We call system H a *generalized optimal dynamical system*. If, in particular, H is a function, we call it an *optimal dynamic system*. In what follows, we will demonstrate that H can in fact be a chaotic optimal dynamical system. For the characterization of chaotic motion, we will use the following result due to Lasota and Yorke (1974) and Li and Yorke (1978).

Proposition2: *Let f be a function on a closed interval I into itself satisfying that it is continuously twice differentiable everywhere except one point* $b \in I$, *and that there is an* $\varepsilon > 0$ *such that*

$|f'(x)| > 1 + \varepsilon$ *for any* x *at which* f'

exists (expansive and unimodal). Then, there is a unique invariant measure on I, μ, *that is ergodic with respect to f and absolutely continuous with respect to the Lebesgue measure.*

This implies that almost every trajectory following an expansive and unimodal dynamical system behaves as if it were stochastic. The tent map is a well-known example of an expansive and unimodal system.

3. THEOREMS

In order to obtain our result, we set

(3.1) $a_{11} / a_{21} = 1$,

and adopt the following notation.

(3.2) $\mu = 1 / a_{22}$;

(3.3) $\gamma = a_{12} / a_{22} - 1$.

The candidate for our chaotic optimal dynamical system is the following function:

(3.4)

$$h(k_{t-1}) = \begin{cases} \mu k_{t-1} & \text{if} \quad 0 \le k_{t-1} \le 1/(\gamma+1) \\ -(\mu/\gamma)(k_{t-1} - 1) & \text{if} \quad 1/(\gamma+1) \le k_{t-1} \le 1. \end{cases}$$

Under the assumption of $\mu/(1+\gamma) \le 1$, function h maps the unit interval [0, 1] into itself. For all the practical purposes, therefore, we may restrict h to the closed interval $I = [0, \mu/(1+\gamma)]$ and treat it as a function on I onto itself. Our main result can be summarized by the following two theorems.

Theorem 1: *Let* $a_{11} / a_{21} = 1, \mu = 1 / a_{22}$ *and* $\gamma = a_{12} / a_{22} - 1$. *Moreover, let* h_I *be the function (3.4) restricted to interval* $I = [0, \mu/(\gamma+1)]$. *Suppose that parameters* μ, ρ *and* γ *satisfy*

(3.5) $0 < \rho < 1$, $\rho\mu > 1$ and $\mu \le \gamma + 1$

Then, on interval I, the generalized optimal dynamical system H(k) coincides with function h_I *if one of the following two conditions are satisfied.*

Condition A: $\mu \le \gamma$;

Condition B:

$$\gamma < \mu \leq \min\left\{\frac{\gamma+\sqrt{\gamma^2+4\gamma}}{2},\ \frac{-1+\sqrt{1+4\gamma}}{2\rho}\right\}.$$

In general, the graph of function h_l is tent-shaped. It can be illustrated by the kinked segment OPQ in Figure 1, 2 and 3. Segment OP lies on the ray from the origin with the slope $\mu>0$ while segment PQ lies on the line through point $(1/(1+\gamma),\ \mu/(1+\gamma))$ with slope $-\mu/\gamma<0$.

In the case in which Condition A is satisfied, function h_l is unimodal but not

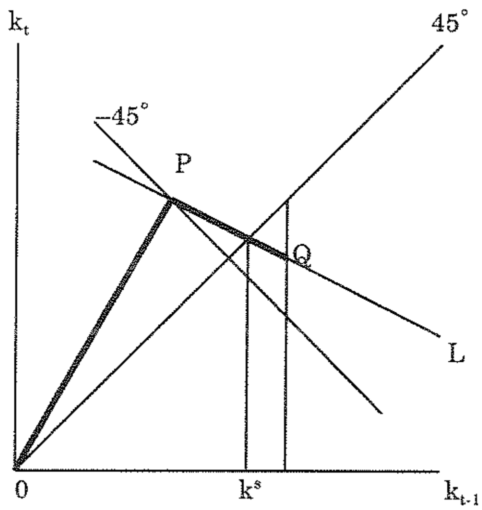

Figure 1

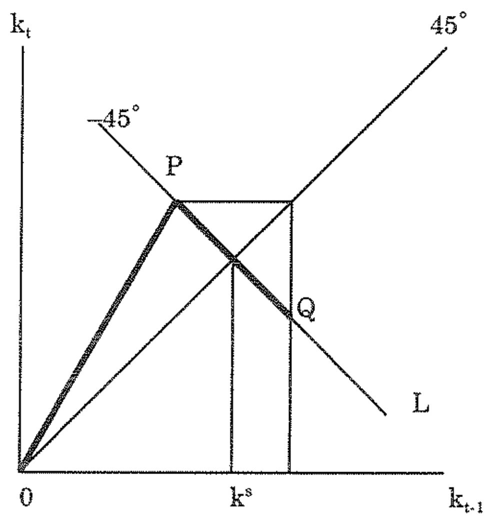

Figure 2

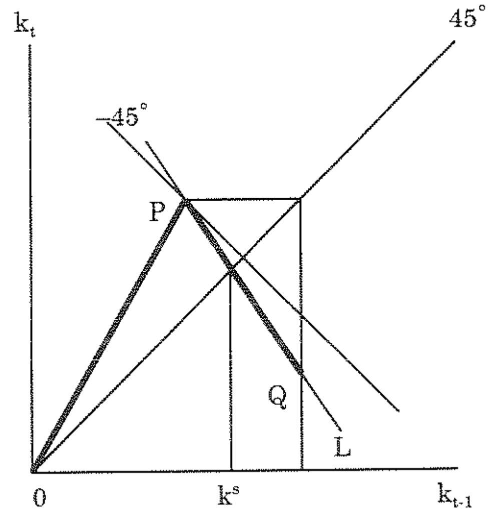

Figure 3

expansive; in this case, given (3.5),

(3.6) $0<\rho<1$, $\rho\mu>1$ and $\mu\leq\gamma$.

If $-\mu/\gamma>-1$, as is shown in Figure 1, optimal dynamical system h_t is globally stable; along any solution to (2.1) with $k>0$, k_t converges to the non-zero fixed point of system h, k^s. If $-\mu/\gamma=-1$, as is shown in Figure 2, along any solution to (2.1) with $k\neq 0, k^s$, k_t goes into a limit cycle of period 2.

If Condition B is satisfied, as is shown in Figure 3, function h is unimodal and expansive. In this sense, by Proposition1, the optimal dynamical system is chaotic. The existence of chaotic solutions is guaranteed by the next result together with Theorem 1.

Theorem 2: *The set of parameters satisfying (3.5) and Condition B of Theorem 1 at the same time is non-empty.*

An intuitive explanation of our result is as follows. First, notice that if both constraints (i) and (ii) of (2.1) are satisfied with equality, c_t and k_t are uniquely determined for each given k_{t-1}. In Figure 1, 2 and 3, line L captures the relationship between k_{t-1} and k_t in this case, i.e., in the case in which constraints (i) and (ii) are both binding.

This leads to the first intuition that there must exist cases in which along the optimal solution to (2.1), these constraints should be both binding. In those cases, the graph of the optimal dynamical system lies on line L. Because each variable must be non-negative ($c_t\geq 0, k_t\geq 0$ and $k_{t-1}\geq 0$), it may be demonstrated that (k_{t-1}, k_t) cannot lie above point P if (c_t, k_t) satisfies constraints (i) and (ii) for a given $k_{t-1}\geq 0$. In short, only if $k_{t-1}\geq 1/(1+\gamma)$, the graph of the optimal dynamical system can lie on line L.

This leads to the second intuition that if $k_{t-1} < 1/(1+\gamma)$, the graph of the optimal dynamical system must be at the position closest to line L. This position is given by segment OP; i.e., for a given $k_{t-1} < 1/(1+\gamma)$, the maximum k_t such that (c_t, k_t) satisfies constraints (i) and (ii) together with $c_t \geq 0$ and $k_t \geq 0$ appears on segment OP.

It follows from these intuitions that it is possible for the tent-shaped graph OPQ is the optimal dynamical system, solving (2.1). As Theorems 1 and 2 demonstrate, these intuitions do hold true under certain conditions but not unconditionally.

4. EXTENSIONS

Discount factor ρ governs the speed of decay in the value of argument c_t of the objective function, $\sum_{t=1}^{\infty} \rho^t c_t$. It is not the case that for any ρ, $0<\rho<1$, (ρ, μ, γ) exists that give rise to a chaotic optimal dynamical system (i.e., satisfies (3.5) and Condition B of Theorem 1). From the viewpoints of various applications, it is important to see how slow the speed of decay (i.e., how close to 1 discount factor ρ) can be for an optimal dynamical system to be chaotic.

Our result guarantees the existence of a chaotic optimal dynamical system for values of discount factor ρ up to 0.5. In other words, the least upper bound of ρ such that (ρ, μ, γ) satisfies (3.5) together with Condition B of Theorem 1 is

(4.1) $\rho^* = 0.5$.

This upper bound poses a serve limitation on the economic application of chaotic optimal dynamics. For example, dynamic LP problem (2.1) may be interpreted as a model of capital accumulation, in which c_t and k_t may be thought of as representing, respectively, levels of consumption and capital stock. Under such an interpretation, ρ may be thought of as determining the length of an individual period of the model. It is generally considered that ρ is around 0.95 in economic models in which the length of an individual period is one year. If $\rho < 0.5$, therefore, the length of a period becomes about a half decade. In other words, the economic application of chaotic optimal dynamics is limited to a model in which the length of a single period is assumed to be more than a half decade.

It is important to note that ρ^* is not the least upper bound of discount factors with which a chaotic optimal dynamical system can appear. By using essentially the same model as LP problem (2.1), Nishimura and Yano (1995) demonstrate that no matter how close ρ is to 1, it is possible to choose (μ, γ) in such way that the optimal dynamical system can be chaotic.[2] That result is based on the assumption that point P in Figures 1, 2 and 3 is a cyclical point of dynamical system h_l.

In other words, neither the result of Nishimura and Yano (1995) nor that reported in this note provides a complete characterization for dynamical system h_l to be the solution to problem (2.1). It is an important subject left for future research to derive a necessary and sufficient condition on parameters (ρ, μ, γ) under which dynamical system h_l is optimal. Such a characterization would provide a better understanding on the possibility of chaotic optimal dynamics.

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[2] Also see Nishimura, Sorger and Yano (1994), which demonstrate a similar result in a different setting.